Vector functions

Questions

Question 1 (Adapted from an exercise on the current homework)**.** Consider the trajectory

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

(a) Find the acceleration $\mathbf{a}(t)$.
(b) Decompose the acceleration as the sum of a tangential vector and a normal vector.

Note that the actual textbook problem only asks for the tangential and normal components of acceleration (which are scalars) and you could compute them directly using the formulas in the book skipping (a).

Question 2. A cannon has the ability to fire a projectile at a fixed speed v but at an adjustable angle θ measured with respect to flat ground. If the goal is to have the projectile land as far away as possible, what is the optimal angle θ?

Assume that the acceleration experienced by the particle at all times is $\langle 0, -g \rangle$ where g is a constant. Here x is the horizontal coordinate and y is the vertical coordinate.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) The acceleration $\mathbf{a}(t)$ is computed as $\mathbf{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$.

(b) One way to do this problem is to compute the scalars a_T and a_N, and then to give the answers $a_T\mathbf{T}$ and $a_N\mathbf{N}$. Alternatively, we can observe that $a_T\mathbf{T}$ is just the projection of $\mathbf{a}$ onto $\mathbf{r}'$:

$$\frac{\langle -\cos t, -\sin t, 0\rangle \cdot \langle -\sin t, \cos t, 1\rangle}{\langle -\sin t, \cos t, 1\rangle \cdot \langle -\sin t, \cos t, 1\rangle}\langle -\sin t, \cos t, 1\rangle = \mathbf{0}.$$

Hence the tangential vector is $\mathbf{0}$, meaning the normal part is the entirety of $\mathbf{a} = \langle -\cos t, -\sin t, 0\rangle$.

Question 2. The initial velocity is $\langle v\cos\theta, v\sin\theta$. The trajectory can be computed, by integration, to be

$$\mathbf{r}(t) = \langle tv\cos\theta, tv\sin\theta - \frac{1}{2}gt^2\rangle.$$

The projectile hits the ground when $tv\sin\theta - \frac{1}{2}gt^2 = 0$, i.e. $t = (2v\sin\theta)/g$. ($t = 0$ is also a solution, but that's when the projectile was launched, not the time of impact.)

At that time, the horizontal displacement is $(2v^2\sin\theta\cos\theta)/g = (v^2\sin(2\theta))/g$. The θ which maximizes this quantity is $\theta = \pi/4$.